

Report on a new type - mixed V and U binomials' recurrence.

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Summary

In [1, 2011] we had proposed $H(x)$ -binomials' recurrence formula appointed by Ward-Horadam $H(x) = \langle H_n(x) \rangle_{n \geq 0}$ functions' sequence i.e. any functions' sequence solution of the second order recurrence with functions' coefficients.

Then we had delivered a new type $H(x)$ -binomials' "mixed" recurrence for primordial case. Here we report on this new type - mixed V and U binomials' recurrence - in brief.

For the abundant list of references on the generalized binomials recurrences subject we refer the reader to [1, 2011].

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1 Sketchy introduction to the history of the subject

Up to our knowledge it was François Édouard Anatole Lucas in [2, 1878] who was the first who had *not only* defined *fibonomial* coefficients as stated in [3, 1989] by Donald Ervin Knuth and Herbert Saul Wilf but who was the first who had defined $U_n \equiv n_{p,q}$ -binomial coefficients $\binom{n}{k}_U \equiv \binom{n}{k}_{p,q}$ and had derived a recurrence for them: see page 27, formula (58) [2, 1878]. Then - referring to Lucas - the investigation relative to divisibility properties of relevant number Lucas sequences D , S as well as numbers' D - binomials and numbers' D - multinomials was continued in [4, 1913] by Robert Daniel Carmichel; see pp. 30,35 and 40 in [4, 1913] for $U \equiv D = \langle D_n \rangle_{n \geq 0}$ and $\binom{n}{k_1, k_2, \dots, k_s}_D$ - respectively. Note there also formulas (10), (11) and (13) which might perhaps serve to derive explicit untangled form of recurrence for the *V- binomial coefficients* $\binom{n}{k}_V \equiv \binom{n}{k}_S$ denoted by primordial Lucas sequence $\langle S_n \rangle_{n \geq 0} = S \equiv V$.

Multinomial coefficients' *recurrences* are not present in that early and other works and up to our knowledge a special case of such appeared at first in [5,

1979] by Anthony G. Shannon. More on that - in what follows after Definition 3?.

1.2. We deliver here - a new $H(x)$ -binomials' recurrence formula appointed by Ward-Horadam $H(x) = \langle H_n(x) \rangle_{n \geq 0}$ field of zero characteristic nonzero valued functions' sequence which comprises for $H \equiv H(x = 1)$ number sequences case - the V -binomials' recurrence formula determined by the primordial Lucas sequence of the second kind $V = \langle V_n \rangle_{n \geq 0}$ [1, 2011] as well as its well elaborated companion fundamental Lucas sequence of the first kind $U = \langle U_n \rangle_{n \geq 0}$ which gives rise in its turn to the U -binomials' recurrence as in [2, 1878] , [6, 1949], [7, 1964], [8, 1969], [9, 1989] or in [3, 1989] and so on.

We do it by following recent applicable work [10, 2009] by Nicolas A. Loehr and Carla D. Savage (see more in [1, 2011]).

This looked for here new $H(x)$ -binomials' overall recurrence formulas -(recall: these should be encompassing V -binomials for primordial Lucas sequence V) - this looked for here recurrence for V is not present neither in [2] nor in [10], nor in [11, 1915] , nor in [12, 1936], nor in [6, 1949]. Neither we find it in all 206 quoted in [1, 2011] references.

Interrogation. Might it be so that such an overall recurrence formula including V does not exist? The method to obtain formulas is more general than actually produced family of formulas? See Theorems 2a and 2b, and Example 1 in what follows.

2 Prerequisites

The Lucas sequence $V = \langle V_n \rangle_{n \geq 0}$ is called the Lucas sequence of the second kind - see: [13, 1977, Part I], or **primordial** - see [14, 1979].

The Lucas sequence $U = \langle U_n \rangle_{n \geq 0}$ is called the Lucas sequence of the first kind - see: [13, 1977, Part I], or **fundamental** - see p. 38 in [6, 1949] or see [5, 1979] and [14, 1979].

In the sequel - following [1, 2011] - we shall deliver the looked for recurrences for H -binomial coefficients $\binom{n}{k}_H$ determined by the Ward-Horadam sequence H - defined below. It appears to be mixed V and U binomials' recurrence in the primordial case $H = V$.

In compliance with Edouard Lucas' [2, 1878] and twenty, twenty first century p, q -people's notation [1, 2011] we at first review here in brief the general second order recurrence; (compare this review with the recent "Ward-Horadam" peoples' paper [15, 2009] by Tian-Xiao He and Peter Jau-Shyong Shiue or earlier p, q -papers [16, 2001] by Zhi-Wei Sun, Hong Hu, J.-X. Liu and [17, 2001] by Hong Hu and Zhi-Wei Sun). And with respect to notation: If in [2, 1878] François Édouard Anatole Lucas had been used $a = \mathbf{p}$ and $b = \mathbf{q}$ notation, he

would be perhaps at first glance notified and recognized as a Great Grandfather of all the (p, q) - people. Let us start then introducing reconciling and matched denotations and nomenclature.

$$(1) \quad H_{n+2} = P \cdot H_{n+1} - Q \cdot H_n, \quad n \geq 0 \text{ and } H_0 = a, H_1 = b.$$

which is sometimes being written in $\langle P, -Q \rangle \mapsto \langle s, t \rangle$ notation.

$$(2) \quad H_{n+2} = s \cdot H_{n+1} + t \cdot H_n, \quad n \geq 0 \text{ and } H_0 = a, H_1 = b.$$

We exclude the cases when (2) is recurrence of the first order, therefore we assume that the roots p, q of (5) are distinct $p \neq q$ and $\frac{p}{q}$ is not the root of unity. We shall come back to this finally while formulating this note observation named Theorem 2a.

Simultaneously and collaterally we mnemonically pre adjust the starting point to discuss the $F(x)$ polynomials' case via - if entitled - antecedent " \mapsto action": $H \mapsto H(x)$, $s \mapsto s(x)$, $t \mapsto t(x)$, etc.

$$(3) \quad H_{n+2}(x) = s(x) \cdot H_{n+1}(x) + t(x) \cdot H_n, \quad n \geq 0, \quad H_0 = a(x), H_1 = b(x).$$

enabling recovering explicit formulas also for sequences of polynomials correspondingly generated by the above linear recurrence of order 2 - with Pafnuty Lvovich Tchebysheff polynomials and the generalized Gegenbauer-Humbert polynomials included. See for example Proposition 2.7 in the recent "Ward-Horadam peoples'" paper [15, 2009] by Tian-Xiao He and Peter Jau-Shyong Shiue.

The general solution of (1): $H(a, b; P, Q) = \langle H_n \rangle_{n \geq 0}$ is being called throughout this note - **Ward-Horadam number's sequence** [1, 2011].

The Lucas U -binomial coefficients $\binom{n}{k}_U \equiv \binom{n}{k}_{p,q}$ are then defined as follows: ([2, 1878], [11, 1915], [12, 1936], [6, 1949], [7, 1964], [8, 1969] etc.)

Definition 1 Let U be as in [2, 1878] i.e $U_n \equiv n_{p,q}$ then U -binomial coefficients for any $n, k \in \mathbb{N} \cup \{0\}$ are defined as follows

$$(4) \quad \binom{n}{k}_U \equiv \binom{n}{k}_{p,q} = \frac{n_{p,q}!}{k_{p,q}! \cdot (n-k)_{p,q}!} = \frac{n_{p,q}^k}{k_{p,q}!}$$

where $n_{p,q}! = n_{p,q} \cdot (n-1)_{p,q} \cdot \dots \cdot 1_{p,q}$ and $n_{p,q}^k = n_{p,q} \cdot (n-1)_{p,q} \cdot \dots \cdot (n-k+1)_{p,q}$ and $\binom{n}{k}_U = 0$ for $k > n$.

Definition 2 Let V be as in [2, 1878] i.e $V_n = p^n + q^n$, hence $V_0 = 2$ and $V_1 = p + q = s$. Then V -binomial coefficients for any $n, k \in \mathbb{N} \cup \{0\}$ are defined as follows

$$(5) \quad \binom{n}{k}_V = \frac{V_n!}{V_k! \cdot V_{(n-k)}!} = \frac{V_n^k}{V_k!}$$

where $V_n! = V_n \cdot V_{n-1} \cdot \dots \cdot V_1$ and $V_n^k = V_n \cdot V_{n-1} \cdot \dots \cdot V_{n-k+1}$ and $\binom{n}{k}_V = 0$ for $k > n$.

One automatically generalizes number F -binomial coefficients' array to functions $F(x)$ -**multinomial** coefficients' array (see [20, 2004] and references to umbral calculus therein) while for *number sequences* $F = F(x = 1)$ the F -multinomial coefficients see p. 40 in [4, 1913] by Robert Daniel Carmichel, see [12, 1936] by Morgan Ward and [8, 1969] by Henri W. Gould; (see more in [1, 2011]).

$\psi(x)$ -**multinomial coefficients**.

Considerations in [5, 1979] by Anthony G. Shannon and an application in [18, 2001] by Thomas M. Richardson as well as relevance to umbral calculus [19, 2003] or [21, 2001] constitute motivating circumstances for considering now *functions'* $F(x)$ -**binomial** and $F(x)$ -**multinomial** coefficients. This is the case ($F(x) = \psi(x)$) for example in [20, 2004] **wherein we read:**

$$(x_1 + {}_\psi x_2 + {}_\psi \dots + {}_\psi x_k)^n = \sum_{\substack{s_1, \dots, s_k = 0 \\ s_1 + s_2 + \dots + s_k = n}}^n \binom{n}{s_1, \dots, s_k}_\psi x_1^{s_1} \dots x_k^{s_k}$$

where

$$\binom{n}{s_1, \dots, s_k}_\psi = \frac{n_\psi!}{(s_1)_\psi! \dots (s_k)_\psi!}.$$

Here above the *u-multinomial* number sequence formula from [12, 1936] by Morgan Ward is extended mnemonically to $\psi(x)$ function sequence definition of shifting x arguments formula written in Kwaśniewski upside-down notation (see for example [19, 2003], [21, 2001], [22, 2002], [23, 2005] for more ad this notation).

The above formulas introduced in case of number sequences in [12, 1936] by Morgan Ward were recalled from [12, 1936] by Alwyn F. Horadam and Anthony G. Shannon in [24, 1976] were the Ward Calculus of sequences framework (including umbral derivative and corresponding exponent) was used to enunciate two types of Ward's Staudt-Clausen theorems pertinent to this general calculus of number sequences.

3 $H(x)$ -binomial and mixed binomial coefficients' recurrence

3.1. Let us recall convention resulting from (3).

Recall. The general solution of (3):

$$H(x) \equiv H(a(x), b(x); s(x), t(x)) = \langle H_n(x) \rangle_{n \geq 0}$$

is being called throughout this paper - **Ward-Horadam functions' sequence**.

Theorem 1 *Let us admit shortly the abbreviations: $g_k(r, s)(x) = g_k(r, s)$, $k = 1, 2$. Let $s, r > 0$. Let $F(x)$ be any zero characteristic field nonzero valued functions' sequence ($F_n(x) \neq 0$). Then*

$$(6) \quad \binom{r+s}{r, s}_{F(x)} = g_1(r, s) \cdot \binom{r+s-1}{r-1, s}_{F(x)} + g_2(r, s) \cdot \binom{r+s-1}{r, s-1}_{F(x)}$$

where $\binom{r}{r, 0}_{F(x)} = \binom{s}{0, s}_{F(x)} = 1$ and

$$(7) \quad F(x)_{r+s} = g_1(r, s) \cdot F(x)_r + g_2(r, s) \cdot F(x)_s.$$

are equivalent.

An on the way historical note Donald Ervin Knuth and Herbert Saul Wilf in [3, 1989] stated that Fibonomial coefficients and the recurrent relations for them appeared already in 1878 Lucas work (see: formula (58) in [2, 1878] p. 27 ; for U -binomials which "Fibonomials" are special case of). More over on this very p. 27 Lucas formulated a conclusion from his (58) formula which may be stated in notation of this paper formula (2) as follows: if $s, t \in \mathbb{Z}$ and $H_0 = 0$, $H_1 = 1$ then $H \equiv U$ and $\binom{n}{k}_U \equiv \binom{n}{k}_{n_{p,q}} \in \mathbb{Z}$.

Consult for that also the next century references: [25, 1910] by Paul Gustav Heinrich Bachmann or later on - [4, 1913] by Robert Daniel Carmichel [p. 40] or [6, 1949] by Dov Jarden and Theodor Motzkin where in all quoted positions it was also shown that $n_{p,q}$ - binomial coefficients are integers - for p and q representing distinct roots of the characteristic equation for (3) - see below.

It seems to be the right place now to underline that the *addition formulas* for Lucas sequences below with respective hyperbolic trigonometry formulas and also consequently U -binomials' recurrence formulas - stem from commutative ring R identity: $(x - y) \cdot (x + y) \equiv x^2 - y^2, x, y \in R$.

Indeed. Recall the characteristic equation notation: $z^2 = s \cdot z + t$ as is common in many publications with the restriction: p, q roots are distinct. Recall that

$p + q = s$ and $p \cdot q = -t$. Let $\Delta = s^2 - 4t$. Then $\Delta = (p - q)^2$. Hence we have as in [2, 1878]

$$(8) \quad 2 \cdot U_{r+s} = U_r V_s + U_s V_r, \quad 2 \cdot V_{r+s} = V_r V_s + \Delta \cdot U_s U_r.$$

Taking here into account the **U-addition formula** i.e. the first of two trigonometric-like L -addition formulas (42) from [2, 1878] ($L = U, V$) one readily recognizes that the U -binomial recurrence from the Corollary 18 in [10, 2009] is the U -binomial recurrence (58) [2, 1878] which may be rewritten after François Édouard Anatole Lucas in multinomial notation and stated as follows: *according to the Theorem 2a below the following is true:*

$$2 \cdot U_{r+s} = U_r V_s + U_s V_r$$

is equivalent to

$$(9) \quad 2 \cdot \binom{r+s}{r, s}_{n_{p,q}} = V_s \cdot \binom{r+s-1}{r-1, s}_{n_{p,q}} + V_r \cdot \binom{r+s-1}{r, s-1}_{n_{p,q}}.$$

However there is no companion V -binomial recurrence i.e. for $\binom{r+s}{r, s}_V$ neither in [2, 1878] nor in [10, 2009] as well as all other quoted papers - up to knowledge of this note author.

The End of the on the way historical note.

The looked for $H(x)$ -binomial recurrence (6) accompanied by (7) might be then given right now in the form of (10) adapted to - Ward-Lucas functions' sequence case notation while keeping in mind that of course the expressions for $h_k(r, s)(x)$, $k = 1, 2$ below are designated by this $F(x) = H(x)$ choice and as a matter of fact are appointed by the recurrence (3).

For the sake of commodity to write down an observation to be next let us admit shortly the abbreviations: $h_k(r, s)(x) = h_k(r, s) = h_k$, $k = 1, 2$. Then for $H(x)$ of the form (21?) we evidently have what follows.

Theorem 2a.

$$(10) \quad \binom{r+s}{r, s}_{H(x)} = h_1(r, s) \binom{r+s-1}{r-1, s}_{H(x)} + h_2(r, s) \binom{r+s-1}{r, s-1}_{H(x)},$$

where $\binom{r}{r, 0}_{H(x)} = \binom{s}{0, s}_{H(x)} = 1$, *is equivalent to*

$$(11) \quad H_{r+s}(x) = h_1(r, s)H_r(x) + h_2(r, s)H_s(x).$$

where $H_n(x)$ is explicitly given by (21) and (22) formulas in [1, 2011].

The end of the Theorem 2a.

Let us now come back to consider the case of V -binomials recurrence. For that to do carefully let us at first make precise the main item. $H(x) = \langle H_n(x) \rangle_{n \geq 0}$ is considered as a solution of **second** order recurrence (3) with peculiar case of (3) becoming the first order recurrence **excluded**. This is equivalent to say that $0 \neq p \neq q \neq 0$ and $A \neq 0 \neq B$ (compare with [26, 1974] by Anthony J. W. Hilton), where for notation convenience we shall again use awhile shortcuts for (3):

$$a(x) \equiv a, b(x) \equiv b, s(x) \equiv s, t(x) \equiv t,$$

$$p(x) \cdot q(x) = -t(x) \equiv p \cdot q = -t,$$

$$H(x) = H(A(x), B(x), p(x), q(x)) = H(A, B) = A \cdot H(1, 1;),$$

$$H(1, 1) = V(x) = V = (p-q)U(1, -1), U = U(x) = U(1, 1), U(A, B) = U(A, B, p, q)$$

referring to this note (23) and the next to (23) abbreviations:

$$H(x) \equiv H(a(x), b(x); s(x), t(x)) \equiv H(A(x), B(x); s(x), t(x)),$$

$$H_n(x) \equiv H_n(a(x), b(x); s(x), t(x)),$$

$$H_n(x) = H_n(A, B) = A \cdot p^n + B \cdot q^n \quad H_n(1, 1) = V_n(x) = V_n.$$

Now although $H(A, B) = (p-q) \cdot U(A, -B)$ the corresponding recurrences are different. Recall and then compare corresponding recurrences:

The identity

$$(p-q) \cdot (p^{r+s} - q^{r+s}) \equiv (p^{s+1} - q^{s+1}) \cdot (p^r - q^r) - p \cdot q(p^{r-1} - q^{r-1}) \cdot (p^s - q^s)$$

due to $p \cdot q = -t$ is *equivalent to*

$$U_{r+s}(p, q) = U_{s+1} \cdot U_r(p, q) + t \cdot U_{r-1} \cdot U_s(p, q)$$

(combinatorial derivation - see [27, 1999],[28, 1999]). This recurrence in its turn *is equivalent to*

$$\binom{r+s}{r, s}_U = U_{s+1} \cdot \binom{r+s-1}{r-1, s}_U + t \cdot U_{r-1} \cdot \binom{r+s-1}{r, s-1}_U.$$

Similarly *the identity*

$$(p-q) \cdot (p^{r+s} + q^{r+s}) \equiv (p^{s+1} - q^{s+1}) \cdot (p^r + q^r) - p \cdot q(p^{r-1} + q^{r-1}) \cdot (p^s - q^s)$$

due to $p \cdot q = -t$ is *equivalent to*

$$(12) \quad V_{r+s}(p, q) = U_{s+1} \cdot V_r(p, q) + t \cdot V_{r-1} \cdot U_s(p, q)$$

For combinatorial interpretation derivation of the above for Fibonacci and Lucas sequences ($t = 1 = s$) see [27, 1999], [28, 1999] and then see for more [?, 2003] also by Arthur T. Benjamin and Jennifer J. Quinn. A fundamental progress in combinatorial interpretation of generalized binomials was made in [29, 2009] by Bruce E. Sagan and Carla D. Savage.

Definition 3 Let $\{n(x)\} \equiv U_n$. $\langle n(x) \rangle \equiv V_n = p^n(x) + q^n(x)$, hence $V_0 = 2$ and $V_1 = p + q = s(x)$. Let $(p(x) - q(x)) \cdot U_n = p^n(x) - q^n(x)$, hence $U_0 = 0$ and $V_1 = 1$; (roots are distinct). Then V -mixed- U binomial coefficients for any $r, s \in \mathbb{N} \cup \{0\}$; are defined as follows

$$(13) \quad \binom{r+s}{r, s}_{\langle \cdot \rangle / \{ \cdot \}} = \frac{V_{r+s}!}{V_r! \cdot U_s!},$$

$$\binom{n}{k}_{\langle \cdot \rangle / \{ \cdot \}} = 0 \text{ for } k > n \text{ and } \binom{n}{0}_{\langle \cdot \rangle / \{ \cdot \}} = 1.$$

$$\text{Note that: } \binom{r+s}{r, s}_{\langle \cdot \rangle / \{ \cdot \}} \neq \binom{r+s}{s, r}_{\langle \cdot \rangle / \{ \cdot \}}.$$

$$\text{Note that: } \binom{r+s}{r, s}_{\langle \cdot \rangle / \{ \cdot \}} \neq \binom{r+s}{r, s}_{\{ \cdot \} / \langle \cdot \rangle}$$

Theorem 2b. The recurrence (12) is equivalent to

$$(14) \quad \binom{r+s}{r, s}_{\langle \cdot \rangle / \{ \cdot \}} = U_{s+1} \cdot \binom{r+s-1}{r-1, s}_{\langle \cdot \rangle / \{ \cdot \}} + t \cdot U_s \cdot \binom{r+s-1}{r, s-1}_{\langle \cdot \rangle / \{ \cdot \}}.$$

Example 1. Application of Theorems 1, 2a and the Theorem 2b method. Recall the characteristic equation notation: $z^2 = z^2 = s \cdot z + t$ with the restriction that p, q roots are distinct. Recall that $p+q = s$ and $p \cdot q = -t$. Let $\Delta = s^2 - 4t$. Then $\Delta = (p - q)^2$. Hence we have as in [2, 1878]

$$2 \cdot U_{r+s} = U_r V_s + U_s V_r, \quad 2 \cdot V_{r+s} = V_r V_s + \Delta \cdot U_s U_r.$$

Consider **V-addition formula** i.e. the second of two trigonometric-like addition formulas (42) from [2, 1878]. One readily recognizes that *according to the Theorem 2b the following is true: $2 \cdot V_{r+s} = V_r V_s + \Delta \cdot U_s U_r$ is equivalent to*

$$(15) \quad 2 \cdot \binom{r+s}{r, s}_{\langle \cdot \rangle / \{ \cdot \}} = V_s \cdot \binom{r+s-1}{r-1, s}_{\langle \cdot \rangle / \{ \cdot \}} + \Delta \cdot U_r \cdot \binom{r+s-1}{r, s-1}_{\langle \cdot \rangle / \{ \cdot \}}.$$

The effort to discover combinatorial interpretation of mixed binomials is part of a dream of a forthcoming "in statu nascendi" note.

s for mixed $F(x)$ -**multinomial** coefficients' arrays - there are many kinds of them. This is also a part of a subject of a forthcoming "in statu nascendi" note.

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